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Flavia Bonomo, Ivo Koch, Pablo Torres, Mario Valencia-Pabon. k-tuple chromatic number of the cartesian product of graphs. 2014. hal-01103534

**HAL Id: hal-01103534**

**<https://hal.science/hal-01103534>**

Preprint submitted on 14 Jan 2015

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# $k$ -tuple chromatic number of the cartesian product of graphs \*

Flavia Bonomo<sup>†</sup>   Ivo Koch<sup>‡</sup>   Pablo Torres<sup>§</sup>   Mario Valencia-Pabon<sup>¶</sup>

## Abstract

A  $k$ -tuple coloring of a graph  $G$  assigns a set of  $k$  colors to each vertex of  $G$  such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The  $k$ -tuple chromatic number of  $G$ ,  $\chi_k(G)$ , is the smallest  $t$  so that there is a  $k$ -tuple coloring of  $G$  using  $t$  colors. It is well known that  $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$ . In this paper, we show that there exist graphs  $G$  and  $H$  such that  $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$  for  $k \geq 2$ . Moreover, we also show that there exist graph families such that, for any  $k \geq 1$ , the  $k$ -tuple chromatic number of their cartesian product is equal to the maximum  $k$ -tuple chromatic number of its factors.

**Keywords:**  $k$ -tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

## 1 Introduction

A classic *coloring* of a graph  $G$  is an assignment of colors (or natural numbers) to the vertices of  $G$  such that any two adjacent vertices are assigned different colors. The smallest number  $t$  such that  $G$  admits a coloring with  $t$  colors (a  $t$ -coloring) is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ . Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the  $k$ -tuple coloring introduced independently by Stahl [11] and Bollobás and Thomason [3]. A  $k$ -tuple

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\*Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France), UBACyT Grant 20020130100808BA, CONICET PIP 112-200901-00178 and 112-201201-00450CO and ANPCyT PICT 2012-1324 (Argentina)

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coloring of a graph  $G$  is an assignment of  $k$  colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The  $k$ -tuple coloring problem consists into finding the minimum number of colors in a  $k$ -tuple coloring of a graph  $G$ , which we denote by  $\chi_k(G)$ . The *cartesian product*  $G \square H$  of two graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$ , two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}. \quad (1)$$

The identity (1) admits a simple proof first given by Sabidussi [10]. The *Kneser graph*  $K(m, n)$  has as vertices all  $n$ -element subsets of the set  $[m] = \{1, \dots, m\}$  and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that  $m \geq 2n$ , otherwise  $K(m, n)$  has no edges. The Kneser graph  $K(5, 2)$  is the well known *Petersen Graph*. Lovász [9] showed that  $\chi(K(m, n)) = m - 2n + 2$ . The value of the  $k$ -tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if  $k = qn - r$  where  $q \geq 0$  and  $0 \leq r < n$ , then  $\chi_k(K(m, n)) = qm - 2r$ . Stahl's conjecture has been confirmed for some values of  $k$ ,  $n$  and  $m$  [11, 12].

An *homomorphism* from a graph  $G$  into a graph  $H$ , denoted by  $G \rightarrow H$ , is an edge-preserving map from  $V(G)$  to  $V(H)$ . It is well known that an ordinary graph coloring of a graph  $G$  with  $m$  colors is an homomorphism from  $G$  into the complete graph  $K_m$ . Similarly, an  $n$ -tuple coloring of a graph  $G$  with  $m$  colors is an homomorphism from  $G$  into the Kneser graph  $K(m, n)$ .

A graph  $G$  is said *hom-idempotent* if there is a homomorphism from  $G \square G \rightarrow G$ . We denote by  $G \not\rightarrow H$  if there exists no homomorphism from  $G$  to  $H$ .

The *clique number* of a graph  $G$ , denoted by  $\omega(G)$ , is the maximum size of a *clique* in  $G$  (i.e., a complete subgraph of  $G$ ). Clearly, for any graphs  $G$  and  $H$ , we have that  $\chi(G) \geq \omega(G)$  (and so,  $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$ ) and, if there is an homomorphism from  $G$  to  $H$  then,  $\chi(G) \leq \chi(H)$  (and so,  $\chi_k(G) \leq \chi_k(H)$ ).

A *stable set*  $S \subseteq V$  is a subset of pairwise non adjacent vertices of  $G$ . The *stability number* of  $G$ , denoted by  $\alpha(G)$ , is the largest cardinality of a stable set in  $G$ . Let  $m \geq 2n$ . An element  $i \in [m]$  is called a *centre* of a stable set  $S$  of the Kneser graph  $K(m, n)$  if it lies in each  $n$ -set in  $S$ .

**Lemma 1** (Erdős-Ko-Rado [4]). *If  $m > 2n$ , then  $\alpha(K(m, n)) = \binom{m-1}{n-1}$ . An independent set of  $K(m, n)$  with size  $\binom{m-1}{n-1}$  has a centre  $i$ , for some  $i \in [m]$ .*

**Lemma 2** (Hilton-Milner [7]). *If  $m \geq 2n$ , then the maximum size of an stable set in  $K(m, n)$  with no centre is equal to  $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$ .*

A graph  $G = (V, E)$  is *vertex transitive* if its automorphism group acts transitively on  $V$ , that is, for any pair of distinct vertices of  $G$  there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs [5].

**Lemma 3** (No-Homomorphism Lemma, Albertson-Collins [1]). *Let  $G, H$  be graphs such that  $H$  is vertex transitive and  $G \rightarrow H$ . Then,  $\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|$ .*

In this paper, we show that equality (1) does not hold in general for  $k$ -tuple colorings of graphs. In fact, we show that for some values of  $k \geq 2$ , there are Kneser graphs  $K(m, n)$  for which  $\chi_k(K(m, n) \square K(m, n)) > \chi_k(K(m, n))$ . Moreover, we also show that there are families of graphs for which equality (1) holds for  $k$ -tuple colorings of graphs for any  $k \geq 1$ . As far as we know, our results are the first ones concerning the  $k$ -tuple chromatic number of cartesian product of graphs.

## 2 Cartesian products of Kneser graphs

**Lemma 4.** *Let  $G$  be a graph and let  $k > 0$ . Then,  $\chi_k(G \square G) \leq k\chi(G)$ .*

*Proof.* Clearly,  $\chi_k(G \square G) \leq k\chi(G \square G)$ . However, by equality (1) we know that  $\chi(G \square G) = \chi(G)$ , and thus the lemma holds.  $\square$

**Corollary 1.**  $\chi_k(K(m, n) \square K(m, n)) \leq k\chi(K(m, n)) = k(m - 2n + 2)$ .

Larose et al. [8] showed that no connected Kneser graph  $K(m, n)$  is hom-idempotent, that is, for any  $m > 2n$ , there is no homomorphism from  $K(m, n) \square K(m, n)$  to  $K(m, n)$ .

**Lemma 5** ([8]). *Let  $m > 2n$ . Then,  $K(m, n) \square K(m, n) \not\rightarrow K(m, n)$ .*

Concerning the  $k$ -tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

**Lemma 6** ([11]). *If  $1 \leq k \leq n$ , then  $\chi_k(K(m, n)) = m - 2(n - k)$ .*

**Lemma 7** ([11]).  $\chi_k(K(2n + 1, n)) = 2k + 1 + \lfloor \frac{k-1}{n} \rfloor$ , for  $k > 0$ .

**Lemma 8** ([11]).  $\chi_{rn}(K(m, n)) = rm$ , for  $r > 0$  and  $m \geq 2n$ .

By using Lemma 8 we have the following result.

**Lemma 9.** *Let  $m > 2n$ . Then,  $\chi_n(K(m, n) \square K(m, n)) > \chi_n(K(m, n))$ .*

*Proof.* By Lemma 8 when  $r = 1$ , we have that  $\chi_n(K(m, n)) = m$ . If  $\chi_n(K(m, n) \square K(m, n)) = m$ , then there exists an homomorphism from the graph  $K(m, n) \square K(m, n)$  to  $K(m, n)$  which contradicts Lemma 5.  $\square$

By Lemma 6, Lemma 9 and by using Corollary 1, we have that,

**Corollary 2.** *Let  $n \geq 2$ . Then,  $2n+2 \leq \chi_n(K(2n+1, n) \square K(2n+1, n)) \leq 3n$ . In particular, when  $n = 2$ , we have that  $\chi_2(K(5, 2) \square K(5, 2)) = 6$ .*

In the case  $k = 2$  we have by Lemma 9, Lemma 6 and by Corollary 1, the following result.

**Corollary 3.** *Let  $q > 0$ . Then,  $q + 4 \leq \chi_2(K(2n + q, n) \square K(2n + q, n)) \leq 2q + 4$ .*

By Corollary 3, notice that in the case when  $k = n = 2$  and  $q \geq 1$ , we must have that  $\chi_2(K(q + 4, 2) \square K(q + 4, 2)) > q + 4$ , otherwise there is a contradiction with Lemma 5. This provides a gap of one unity between the 2-tuple chromatic number of the graph  $K(q + 4, 2) \square K(q + 4, 2)$  and the graph  $K(q + 4, 2)$ . In the next Lemma 11, we will show that such a gap can be as large as desired. However, first we need to introduce the following.

It is well known that the chromatic index of a complete graph  $K_{2n}$  (i.e. the minimum number of colors needed to color the edges of  $K_{2n}$  such that any two incident edges be assigned different colors) on  $2n$  vertices is equal to  $2n - 1$  (see [2]), where each color class  $i$  (i.e. the subset of pairwise non incident edges colored with color  $i$ ) has size  $n$ . Therefore, using this fact, we obtain the following result.

**Lemma 10.** *Let  $q \geq 1$ . Then, the set of vertices of the Kneser graph  $K(2q + 4, 2)$  can be partitioned into  $2q + 3$  disjoint cliques, each one with size  $q + 2$ .*

*Proof.* Notice that there is a natural bijection between the vertex set of  $K(2q + 4, 2)$  and the edge set of the complete graph  $K_{2q+4}$  with vertex set  $[2q + 4]$ : each vertex  $\{i, j\}$  in  $K(2q + 4, 2)$  is mapped to the edge  $\{i, j\}$  in  $K_{2q+4}$ . Now, there is a  $(2q + 3)$ -edge coloring of  $K_{2q+4}$  where each class color is a set of pairwise non incident edges with size  $q + 2$ . Notice that two edges  $e, e' \in K_{2q+4}$  are non incident edges if and only if  $e \cap e' = \emptyset$ . Therefore, a class color of the edge-coloring of  $K_{2q+4}$  represents a clique of  $K(m, n)$ .  $\square$

**Lemma 11.** *Let  $q > 0$ . Then,  $\chi_2(K(2q + 4, 2) \square K(2q + 4, 2)) \geq 2q + \lceil \frac{2}{3}q \rceil + 5$ .*

*Proof.* First, recall that a stable set  $X$  in  $K(2q + 4, 2)$  has size at most  $2q + 3$  if  $X$  has centre (see Lemma 1) and  $|X| \leq 1 + (2q + 4 - 1) - (2q + 4 - 2 - 1) = 3$  if  $X$  has no centre (see Lemma 2). Besides, by Lemma 10, observe that the vertex set of  $K(2q + 4, 2)$  can be partitioned in  $2q + 3$  sets  $S_1, \dots, S_{2q+3}$  such that each  $S_i$  induces a  $K_{q+2}$  for  $i = 1, \dots, 2q + 3$ . Consider the subgraph  $H_i$  of  $K(2q + 4, 2) \square K(2q + 4, 2)$  induced by  $S_i \times V(K(2q + 4, 2))$  for  $i = 1, \dots, 2q + 3$ . Let  $I$  be a stable set in  $K(2q + 4, 2)$  and  $I_i = I \cap H_i$  for  $i = 1, \dots, 2q + 3$ . Then  $I_i^v = I_i \cap (\{v\} \times V(K(2q + 4, 2)))$  is a stable set in  $K(2q + 4, 2)$  for each  $v \in S_i$  and  $i = 1, \dots, 2q + 3$ .

Now, assume w.l.o.g that  $r$  ( $r \leq q + 2$ ) stable sets  $I_1^1, \dots, I_r^r$  have distinct centre  $j_1, \dots, j_r$ , respectively (the case when two of these stable sets have the same centre can be easily reduced to this case). Let  $W$  be the set of subsets with size two of  $\{j_1, \dots, j_r\}$ . Therefore, for all  $m \in \{1, \dots, r\}$ ,  $I_i^m - W$  has size at most  $2q + 3 - (r - 1) = 2q + 4 - r$  since each centre  $j_m$  belongs to  $r - 1$  elements in  $W$ . Besides, each element of  $W$  belongs to exactly one set  $I_i^m$  for  $m \in \{1, \dots, r\}$ , since  $S_i$  induces a complete subgraph. Then,  $|I_i^1 \cup \dots \cup I_i^r| \leq (\sum_{m=1}^r |I_i^m - W|) + |W| \leq r(2q + 4 - r) + \frac{r(r-1)}{2}$ .

Next, each remaining stable set (if exist)  $I_i^{r+1}, \dots, I_i^{q+2}$  has no centre, then  $|I_i^d| \leq 3$  for all  $d \in \{r+1, \dots, q+2\}$ . Thus,  $|I_i| \leq r(2q+4-r) + \frac{r(r-1)}{2} + 3(q+2-r) = -\frac{r^2}{2} + r(2q+\frac{1}{2}) + 3(q+2)$ . Since the last expression is non decreasing for  $r \in \{1, \dots, q + 2\}$ , we have that

$$|I_i| \leq -\frac{(q+2)^2}{2} + (q+2)(2q+\frac{1}{2}) + 3(q+2) = (q+2)(\frac{3}{2}q + \frac{5}{2}).$$

Therefore,  $|I_i| \leq (q+2)(\frac{3}{2}q + \frac{5}{2})$  for every  $i = 1, \dots, 2q+3$ . Since  $|I| = \sum_{i=1}^{2q+3} |I_i|$ , it follows that  $|I| \leq (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})$  and thus,  $\alpha(K(2q+4, 2) \square K(2q+4, 2)) \leq (2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})$ .

Let  $t < \frac{2q^2+18q+24}{3q+5}$ . Assume that exists a 2-tuple coloring of the graph  $K(2q+4, 2) \square K(2q+4, 2)$  with  $2q+t$  colors. Therefore, there exists an homomorphism from  $K(2q+4, 2) \square K(2q+4, 2)$  to  $K(2q+t, 2)$ .

Now, from the well known no-homomorphism Lemma 3, we have that,

$$\alpha(K(2q+4, 2) \square K(2q+4, 2)) \geq \frac{\alpha(K(2q+t, 2))|V(K(2q+4, 2) \square K(2q+4, 2))|}{|V(K(2q+t, 2))|}.$$

$$\text{Then, } \alpha(K(2q+4, 2) \square K(2q+4, 2)) \geq \frac{(2q+t-1) \cdot (2q+4)^2 (2q+3)^2}{4(2q+t)(2q+t-1)} = \frac{(2q+4)^2 (2q+3)^2}{2(2q+t)}.$$

Let us see that  $\frac{(2q+4)^2 (2q+3)^2}{2(2q+t)}$  is greater than  $(2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})$ , which is a contradiction.

To this end, observe that if  $t < \frac{2q^2+18q+24}{3q+5}$ , then

$$(2q+3)(q+2)(\frac{3}{2}q + \frac{5}{2})(2q+t) = (2q+3)(q+2)((3q^2+5q) + (\frac{3}{2}q + \frac{5}{2})t) < (2q+3)(q+2)((3q^2+5q) + (q^2+9q+12)) = (2q+3)(q+2)(4q^2+14q+12) = \frac{(2q+4)^2 (2q+3)^2}{2}.$$

Therefore,  $\chi_2(K(2q+4, 2) \square K(2q+4, 2)) \geq 2q+t+1$ .

**Claim 1.** Let  $q > 0$  be an integer. Then,  $\frac{2q^2+18q+24}{3q+5}$  is not an integer.

*Proof.* By polynomial division, we have that  $2q^2 + 18q + 24 = (3q+5)(\frac{2}{3}q + \frac{44}{9}) - \frac{4}{9}$ . If  $(2q^2 + 18q + 24)/(3q+5)$  is an integer, then  $k = (\frac{2}{3}q + \frac{44}{9}) - \frac{4}{9(3q+5)}$  is also an integer. Multiplying by 9 both terms in the last equality we have,  $6q+44 = 9k + \frac{4}{3q+5}$ . As  $q > 0$  then,  $0 < \frac{4}{3q+5} < 1$  contradicting the assumption that  $k$  is an integer.

Finally, since  $\frac{2}{3}q + 4 < \frac{2q^2+18q+24}{3q+5}$  and, by previous Claim 1,  $\frac{2q^2+18q+24}{3q+5}$  is not an integer, then  $\lceil \frac{2}{3}q \rceil + 4 < \frac{2q^2+18q+24}{3q+5}$  and thus, we have finally that,

$$2q + \lceil \frac{2}{3}q \rceil + 5 \leq \chi_2(K(2q+4, 2)). \quad \square$$

As a corollary of Lemma 11 and by Corollary 1, we obtain the following result.

**Corollary 4.**  $\chi_2(K(6, 2) \square K(6, 2)) = 8$ .

**Theorem 1.** Let  $k > n$  and let  $t = \chi_k(K(m, n) \square K(m, n))$ , where  $m > 2n$ . Then, either  $t > m + 2(k-n)$  or  $t < m + (k-n)$ .

*Proof.* Suppose that  $m+(k-n) \leq t \leq m+2(k-n)$ . Therefore, there exists an homomorphism  $K(m, n) \square K(m, n) \rightarrow K(t, k)$ . Now, Stahl [11] showed that there is an homomorphism  $K(m, n) \rightarrow K(m-2, n-1)$  whenever  $n > 1$  and  $m \geq 2n$ . Moreover, it's easy to see that there is an homomorphism  $K(m, n) \rightarrow K(m-1, n-1)$ . By applying the former homomorphism  $t - (m + (k - n))$  times to the graph  $K(t, k)$  we obtain an homomorphism  $K(t, k) \rightarrow K(2(m+k-n)-t, 2k+m-n-t)$ . Finally, by applying  $2k+m-t-2n$  times the latter homomorphism to the graph  $K(2(m+k-n)-t, 2k+m-n-t)$  we obtain an homomorphism  $K(2(m+k-n)-t, 2k+m-n-t) \rightarrow K(m, n)$ . Therefore, by homomorphism composition,  $K(m, n) \square K(m, n) \rightarrow K(m, n)$  which contradicts Lemma 5.  $\square$

We can also obtain a lower bound for the  $k$ -tuple chromatic number of the graph  $K(m, n) \square K(m, n)$  in terms of the clique number of  $K(m, n)$ . In fact, notice that  $\omega(K(m, n) \square K(m, n)) = \omega(K(m, n)) = \lfloor \frac{m}{n} \rfloor$ . Thus, we have the following result.

**Theorem 2.** *Let  $k > n$ . Then,  $\chi_k(K(m, n) \square K(m, n)) \geq k\omega(K(m, n)) = k \lfloor \frac{m}{n} \rfloor$ . In particular, if  $n$  divides  $m$  then,  $\chi_k(K(m, n) \square K(m, n)) \geq m + (k - n) \frac{m}{n}$ .*

### 3 Cases where $\chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\}$

**Theorem 3.** *Let  $G$  and  $H$  be graphs such that  $\chi(G) \leq \chi(H) = \omega(H)$ . Then,  $\chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\}$ .*

*Proof.* Let  $t = \omega(H)$  and let  $\{h_1, \dots, h_t\}$  be the vertex set of a maximum clique  $K_t$  in  $H$  with size  $t$ . Clearly,  $\chi_k(G) \leq \chi_k(H) = \chi_k(K_t)$ . Let  $\rho$  be a  $k$ -tuple coloring of  $H$  with  $\chi_k(H)$  colors. By equality (1), there exists a  $t$ -coloring  $f$  of  $G \square H$ . Therefore, the assignment of the  $k$ -set  $\rho(h_{f((a,b))})$  to each vertex  $(a, b)$  in  $G \square H$  defines a  $k$ -tuple coloring of  $G \square H$  with  $\chi_k(K_t)$  colors.  $\square$

Notice that if  $G$  and  $H$  are both bipartite, then  $\chi_k(G \square H) = \chi_k(G) = \chi_k(H)$ . In the case when  $G$  is not a bipartite graph, we have the following results.

An automorphism  $\sigma$  of a graph  $G$  is called a *shift* of  $G$  if  $\{u, \sigma(u)\} \in E(G)$  for each  $u \in V(G)$  [8]. In other words, a shift of  $G$  maps every vertex to one of its neighbors.

**Theorem 4.** *Let  $G$  be a non bipartite graph having a shift  $\sigma \in \text{AUT}(G)$ , and let  $H$  be a bipartite graph. Then,  $\chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\}$ .*

*Proof.* Let  $A \cup B$  be a bipartition of the vertex set of  $H$ . Let  $f$  be a  $k$ -tuple coloring of  $G$  with  $\chi_k(G)$  colors. Clearly,  $\chi_k(G) \geq \chi_k(H)$ . We define a  $k$ -tuple coloring  $\rho$  of  $G \square H$  with  $\chi_k(G)$  colors as follows: for any vertex  $(u, v)$  of  $G \square H$  with  $u \in G$  and  $v \in H$ , define  $\rho((u, v)) = f(u)$  if  $v \in A$ , and  $\rho((u, v)) = f(\sigma(u))$  if  $v \in B$ .  $\square$

We may also deduce the following direct result.

**Theorem 5.** *Let  $G$  be an hom-idempotent graph and let  $H$  be a subgraph of  $G$ . Thus,  $\chi_k(G \square H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$ .*

Let  $A$  be a group and  $S$  a subset of  $A$  that is closed under inverses and does not contain the identity. The *Cayley graph*  $\text{Cay}(A, S)$  is the graph whose vertex set is  $A$ , two vertices  $u, v$  being joined by an edge if  $u^{-1}v \in S$ . If  $a^{-1}Sa = S$  for all  $a \in A$ , then  $\text{Cay}(A, S)$  is called a *normal Cayley graph*.

**Lemma 12** ([6]). *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 5 and Lemma 12 we have the following result.

**Theorem 6.** *Let  $\text{Cay}(A, S)$  be a normal Cayley graph and let  $\text{Cay}(A', S')$  be a subgraph of  $\text{Cay}(A, S)$ , with  $A' \subseteq A$  and  $S' \subseteq S$ . Then,  $\chi_k(\text{Cay}(A, S) \square \text{Cay}(A', S')) = \max\{\chi_k(\text{Cay}(A, S)), \chi_k(\text{Cay}(A', S'))\}$ .*

**Definition 1.** *Let  $G$  be a graph with a shift  $\sigma$ . We define the order of  $\sigma$  as the minimum integer  $i$  such that  $\sigma^i$  is equal to the identity permutation.*

**Theorem 7.** *Let  $G$  be a graph with a shift  $\sigma$  of minimum odd order  $2s + 1$  and let  $C_{2t+1}$  be a cycle graph, where  $t \geq s$ . Then,  $\chi_k(G \square C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$ .*

*Proof.* Let  $\{0, \dots, 2t\}$  be the vertex set of  $C_{2t+1}$ , where for  $0 \leq i \leq 2t$ ,  $\{i, i + 1 \bmod n\} \in E(C_{2t+1})$ . Let  $G_i$  be the  $i^{\text{th}}$  copy of  $G$  in  $G \square C_{2t+1}$ , that is, for each  $0 \leq i \leq 2t$ ,  $G_i = \{(g, i) : g \in G\}$ . Let  $f$  be a  $k$ -tuple coloring of  $G$  with  $\chi_k(G)$  colors. We define a  $k$ -tuple coloring of  $G \square C_{2t+1}$  with  $\chi_k(G)$  colors as follows: let  $\sigma^0$  denotes the identity permutation of the vertices in  $G$ . Now, for  $0 \leq i \leq 2s$ , assign to each vertex  $(u, i) \in G_i$  the  $k$ -tuple  $f(\sigma^i(u))$ . For  $2s + 1 \leq j \leq 2t$ , assign to each vertex  $(u, j) \in G_j$  the  $k$ -tuple  $f(u)$  if  $j$  is odd, otherwise, assign to  $(u, j)$  the  $k$ -tuple  $f(\sigma^1(u))$ . It's not difficult to see that this is in fact a proper  $k$ -tuple coloring of  $G \square C_{2t+1}$ .  $\square$

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